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ON COMPLEMENTED NONABELIAN CHIEF FACTORS OF A FINITE GROUP*

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ABSTRACT

The number of chief factors which are complemented in a finite group G may not be the same in two chief series of G, despite what occurs with the number of frattini chief factors or of chief factors which are complemented by a maximal subgroup of G. In this paper we determine the possible changes on that number. These changes can only occur in a certain type of nonabelian chief factors. All groups considered in this paper are assumed to be finite.

1. G-equivalent G-groups

Recall that a G-group A is a group A with a homomorphism θ: $G \to \mathrm{Aut}\, A$. If there is no confusion we put $a^g = a^{\theta(g)}$, if $a \in A$, $g \in G$. Given a G-group A we have the corresponding semidirect product GA, where the multiplication is given by $g_1a_1 \cdot g_2a_2 = g_1g_2a_1^{g_2}a_2$, $g_i \in G$, $a_i \in A$, $i = 1, 2$. Observe that $\ker\theta = \mathrm{C}_G(A)$.

Two G-groups A and B are said to be G-isomorphic, denoted $A \cong_G B$, if there exists an isomorphism φ: $A \to B$ such that $a^{g\varphi} = a^{\varphi g}$, $a \in A$, $g \in G$. Then we say that φ is a G-isomorphism.

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(1.1) Definition: Let A and B be two G-groups. We say that they are G-**equivalent** and put $A \sim_G B$, if there is an isomorphism $\Phi\colon GA \to GB$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & GA & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Phi} & & \| & & \\ 1 & \longrightarrow & B & \longrightarrow & GB & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Then the extensions $A \rightarrowtail GA \twoheadrightarrow G$ and $B \rightarrowtail GB \twoheadrightarrow G$ are said to be equivalent. They are in fact isomorphic in the category of extensions by G and the pair (φ, Φ) is an isomorphism in the terminology of Gruenberg [5].

It is immediate that this relation is an equivalence relation.

On the other hand, if $\varphi\colon A \to B$ is a G-isomorphism, then $(ga)^\Phi = ga^\varphi$, $g \in G$, $a \in A$, defines an isomorphism $\Phi\colon GA \to GB$ which makes the above diagram commutative. That is, two G-isomorphic G-groups are G-equivalent.

The converse is not true. Consider, for instance, $G = A \times B$, where A and B are isomorphic nonabelian simple groups. Then A and B are G-equivalent, but not G-isomorphic, as $\mathrm{C}_G(A) = B$ and $\mathrm{C}_G(B) = A$.

In this context the nonabelian cohomology is useful. We follow the terminology of Serre in [12]. Given a G-group B and a 1-cocycle $\beta \in \mathrm{Z}^1(G, B)$, then $b^{\eta(g)} = b^{gg^\beta}$ defines a homomorphism $\eta\colon G \to \operatorname{Aut} B$. The corresponding G-group is denoted B_β and called the G-group obtained from B by torsion via β.

(1.2) PROPOSITION: *Let A and B be two G-groups. They are G-equivalent if and only if there exists a 1-cocycle $\beta \in \mathrm{Z}^1(G, B)$ such that $A \cong_G B_\beta$ (i.e. if there exists an isomorphism $\varphi\colon A \to B$ such that $a^{g\varphi} = a^{\varphi g g^\beta}$, $a \in A$, $g \in G$).*

Proof: From right to left: Define $\Phi\colon GA \to GB$ by $(ga)^\Phi = gg^\beta a^\varphi$. The other direction: Define $\beta\colon G \to B$ by $g^\beta = g^{-1}g^\Phi$. ∎

Observe that we are in fact involved with the coupling $\chi\colon G \to \operatorname{Out} A$ induced by the G-group A, i.e. by the homomorphism $\theta\colon G \to \operatorname{Aut} A$. In the case where A is a nonabelian irreducible G-group, we are with the corresponding unique class of equivalence of extensions of the centerless group A by G (e.g. 11.4.10 of [11]).

We have, as an immediate consequence, that two G-equivalent G-groups are similar in the sense of Kovács and Newman [10].

On the other hand, two abelian G-groups are G-equivalent if and only if they are G-isomorphic. (If A is an abelian G-group, then $\operatorname{Inn} A = 1$, hence any torsion –in the above sense– on A is trivial.)

However, as we have seen, for nonabelian G-groups the G-equivalence is strictly weaker than G-isomorphism, and provides some criteria, as the one in the next proposition.

Recall that a chief factor H/K of G is frattini if $H/K \leq \Phi(G/K)$. It is complemented if there exists a subgroup U of G such that $UH = G$ and $U \cap H = K$ (then U is a complement of H/K in G). An abelian chief factor of G is non-frattini if and only if it is complemented, but that is not true for nonabelian chief factors, which are obviously in any case non-frattini.

(1.3) PROPOSITION: *Let H/K be a nonabelian chief factor of G. Then H/K is complemented in G if and only if there exists a G-group B, $B \sim_G H/K$, such that $H \leq \mathrm{C}_G(B)$.*

Proof: Assume that U is a complement of H/K in G. We consider the G-group B whose underlying group is H/K with the action of G defined by

$$\theta\colon G \to \operatorname{Aut} B$$

given by $b^{\theta(g)} = b^u$, and the 1-cocycle

$$\beta\colon G \to B$$

given by $g^\beta = hK$, if $g \in G$, $g = uh$, $u \in U$, $h \in H$. It is immediate that both are well-defined mappings, that θ is a homomorphism and that β is a 1-cocycle.

Let now

$$\varphi\colon H/K \to B$$

given by $(xK)^\varphi = xK$, if $x \in H$. With the above notations, we have

$$(xK)^{g\varphi} = (xK)^g = (xK)^{uh} = [(xK)^u]^{(hK)} = [(xK)^{\varphi\theta(g)}]^{g^\beta},$$

hence $H/K \cong_G B_\beta$ and $H/K \sim_G B$ by (1.2). On the other hand,

$$\mathrm{C}_G(B) = \{uh \mid u \in U,\ h \in H,\ (xK)^u = xK\ \forall x \in H\} = \mathrm{C}_U(H/K)H;$$

in particular, $H \leq \mathrm{C}_G(B)$.

Assume conversely, with (1.2), that we have a G-group B and a G-isomorphism $\varphi\colon B \to (H/K)_\alpha$, where $\alpha \in \mathrm{Z}^1(G, H/K)$, such that $H \leq \mathrm{C}_G(B)$. If $b \in B$, $h \in H$, we have

$$b^\varphi = (b^h)^\varphi = b^{\varphi h h^\alpha},$$

hence $hh^\alpha \in \mathrm{C}_{H/K}(H/K) = 1$, as H/K is a nonabelian chief factor of G. Therefore, $h^\alpha = h^{-1}K$ if $h \in H$.

Take $U = \ker\alpha$. Let $g \in G$ and $g^\alpha = hK$, $h \in H$. We have

$$(gh)^\alpha = g^{\alpha h} h^\alpha = (hK)(h^{-1}K) = 1,$$

hence $G = HU$. If $u \in U \cap H$, then $1 = u^\alpha = u^{-1}K$, and $H \cap U = K$. ∎

Observe that this is precisely a result for nonabelian chief factors: If H/K is an abelian chief factor of G, then $H \leq \mathrm{C}_G(H/K)$, hence with this hypothesis the 'only if' implication in the above proposition is also true if H/K is frattini whereas the converse is false.

Given a group G, the socle $S(G)$ of G is the product of all minimal normal subgroups of G. Recall that a group G is said to be primitive if it has a maximal subgroup with core trivial. The socle of a primitive group G can be either (I) an abelian minimal normal subgroup of G, or (II) a nonabelian minimal normal subgroup of G, or (III) the product of exactly two nonabelian minimal normal subgroups of G. We say then respectively that G is primitive of type I, II or III. Two chief factors are said to be G-related if either they are G-isomorphic between them or to the two minimal normal subgroups of a primitive epimorphic image of type III of G [2], [7]. We set

$$\mathcal{CF}(G) := \{H/K \mid H, K \trianglelefteq G,\ H/K \text{ chief factor of } G\}.$$

Let $\mathrm{I}_G(A) = \{g \in G \mid g \text{ induces an inner automorphism in } A\}$, where A is a G-group. It is immediate from (1.2) that if A and B are two G-equivalent G-groups, then $\mathrm{I}_G(A) = \mathrm{I}_G(B)$.

(1.4) PROPOSITION: *Let $F_1, F_2 \in \mathcal{CF}(G)$. Then the following assertions are equivalent by pairs:*

(1) $F_1 \sim_G F_2$.
(2) *F_1 and F_2 are G-related.*
(3) *Either $F_1 \cong_G F_2$ or there exist $E_i \in \mathcal{CF}(G)$ such that $F_i \cong_G E_i$ $(i = 1, 2)$ and E_1 and E_2 have a common complement in G which is a maximal subgroup of G.*
(4) *Either $F_1 \cong_G F_2$ or there exist $E_i \in \mathcal{CF}(G)$ such that $F_i \cong_G E_i$ $(i = 1, 2)$ and E_1 and E_2 have a common complement in G.*

Proof: We have that two abelian chief factors of G are G-equivalent if and only if they are G-isomorphic if and only if they are G-related. On the other hand, a complement U of an abelian chief factor H/K of G is a maximal subgroup of G and, if $E = \mathrm{core}_G(U)$, then G/E is primitive of type I with socle $S(G/E) = C/E$, where $C = \mathrm{C}_G(H/K)$, and we have $C/E \cong_G H/K$. So we may assume that the factors are nonabelian and not G-isomorphic. Let $F_1 = H/K$, $F_2 = L/M$, where $H, K, L, M \trianglelefteq G$.

(1) $\Longrightarrow$ (2) Set $A = \mathrm{C}_G(H/K)$ and $B = \mathrm{C}_G(L/M)$. Then $A \neq B$. We have $\mathrm{I}_G(H/K) = \mathrm{I}_G(L/M) =: I$. So

$$A/A \cap B \cong_G I/B \cong_G L/M, \qquad B/A \cap B \cong_G I/A \cong_G H/K.$$

We must show that $G/A \cap B$ is primitive of type III. We may assume that $A \cap B = 1$. Then $\mathrm{C}_G(A) = B$ and $\mathrm{C}_G(B) = A$. As we are with an equivalence relation, $B \sim_G A$. Then there exist $\alpha \in \mathrm{Z}^1(G, B)$ and a G-isomorphism $\varphi\colon B \to A_\alpha$. As we see in the proof of (1.3), $U = \ker \alpha$ complements A in G. Let now $b \in B$ and $u \in B \cap U$. We have that

$$b^\varphi = b^{\varphi u} = b^{\varphi u u^\alpha} = b^{u\varphi},$$

hence $b = b^u$. So $u \in \mathrm{C}_G(B) = A$, hence $u = 1$ and $B \cap U = 1$. Hence U is a maximal subgroup of G with trivial core.

(2) $\Longrightarrow$ (3) follows immediately from the definition and (3) $\Longrightarrow$ (4) is trivial.

(4) $\Longrightarrow$ (1) Assume that a subgroup U of G complements both H/K and L/M, where $H/K \cong_G F_1$ and $L/M \cong_G F_2$. So, with $E = \mathrm{core}_G(U)$, U complements EH/E and EL/E. Consider

$$\varphi\colon EL/E \to EH/E \qquad \text{and} \qquad \beta\colon G \to EH/E$$

given, respectively, by $(xE)^\varphi = yE$, if $x \in L$, $y \in H$ and $xy \in U$, and $g^\beta = yE$ if $g \in G$, $y \in H$ and $gy \in U$. Then $\beta \in \mathrm{Z}^1(G, EH/E)$ and $\varphi\colon EL/E \to (EH/E)_\beta$ is a G-isomorphism, that is, $EL/E \sim_G EH/E$. ∎

(In the proof (1) $\Longrightarrow$ (2) it is not sufficient to show that $G/A \cong G/B$ is a primitive group of type II, s. [2] 1.2(b).)

(1.5) Definition: Let A be an irreducible G-group. Put $I = \mathrm{I}_G(A)$. We set

$$\mathrm{D}_G(A) = \bigcap \{R \mid R \leq I, R \trianglelefteq G, A \sim_G I/R, I/R \text{ is non-frattini}\},$$
$$\mathrm{E}_G(A) = \{g \in G \mid g^\alpha = 1 \ \forall \alpha \in \mathrm{Z}^1(G, A)\}.$$

Observe that if $A \sim_G B$, then $\mathrm{D}_G(A) = \mathrm{D}_G(B)$. The quotient $\mathrm{I}_G(A)/\mathrm{D}_G(A)$ is the so-called A-crown of G(a generalization of a concept due to Gaschütz [3]) of interest on some questions of Schunck classes or of classes of groups in general [2], [7]. When A is abelian, in fact an irreducible $\mathbb{F}_l\mathbb{G}$-module, this crown appears in relation to the principal indecomposable modules of the group algebra $\mathbb{F}_l\mathbb{G}$ [1], [6], [9].

If $\alpha \in \mathrm{Z}^1(G,A)$, $\beta \mapsto \alpha\cdot\beta$ defines a bijection between $\mathrm{Z}^1(G,A_\alpha)$ and $\mathrm{Z}^1(G,A)$ [12], so $\mathrm{E}_G(A) = \mathrm{E}_G(B)$ if $A \sim_G B$.

Let A be a G-group. We put $A^G = \mathrm{H}^0(G,A)$. Let $N \trianglelefteq G$. Then A is, by restriction, an N-group and $\mathrm{Z}^1(N,A)$ is a G-set with $n^{(\nu^g)} = (n^{g^{-1}})^{\nu g}$, if $n \in N$, $\nu \in \mathrm{Z}^1(N,A)$, $g \in G$. Recall that $\mathrm{H}^1(G,A) = \mathrm{Z}^1(G,A)/\sim$, where if $\nu,\nu' \in \mathrm{Z}^1(N,A)$, then $\nu \sim \nu'$ if there exists $a \in A$ such that $n^{\nu'} = (a^{-1})^n n^\nu a$. In this case we have

$$n^{\nu'^g} = (n^{g^{-1}})^{\nu' g} = \left((a^{-1})^{n^{g^{-1}}} n^{g^{-1}\nu} a\right)^g = (a^{-1})^{gn} n^{\nu^g} a^g,$$

hence $\nu^g \sim \nu'^g$, that is, $\mathrm{H}^1(N,A)$ is also a G-set.

As for G-modules [5] we have the following

(1.6) LEMMA: *Let*

$$N \rightarrowtail G \twoheadrightarrow G/N$$

be a short exact sequence of groups, where $N \trianglelefteq G$ and the arrows are the canonical inclusion and projection. If A is a G-group, we have the following exact sequences of pointed sets:

$$0 \longrightarrow \mathrm{Z}^1(G/N, A^N) \xrightarrow{\text{inf}} \mathrm{Z}^1(G,A) \xrightarrow{\text{res}} \mathrm{Z}^1(N,A),$$
$$0 \longrightarrow \mathrm{H}^1(G/N, A^N) \xrightarrow{\text{inf}} \mathrm{H}^1(G,A) \xrightarrow{\text{res}} \mathrm{H}^1(N,A)^G,$$

where inf *and* res *denote the corresponding inflation and restriction maps.*

Proof: It is a routine check. ∎

(1.7) THEOREM: *Let A be an irreducible G-group and $N \trianglelefteq G$, $N \le \mathrm{C}_G(A)$. Then the following assertions are equivalent by pairs:*

$$(1) N \le \mathrm{E}_G(A), \quad (2)\, \mathrm{Z}^1(G,A) = \mathrm{Z}^1(G/N,A), \quad (3)\, \mathrm{H}^1(G,A) = \mathrm{H}^1(G/N,A).$$

Proof: If suffices to consider the lemma and note that the inflation is bijective if and only if the restriction is null and that is equivalent to $N \le \ker(\alpha)$ $\forall\alpha \in \mathrm{Z}^1(G,A)$. ∎

As a consequence of [9] we have:

(1.8) COROLLARY: *If A is an abelian irreducible G-group, then* $\mathrm{E}_G(A) = \mathrm{D}_G(A)$.

In the following we are interested in the nonabelian case.

Recall that, if A is a G-group, then $\alpha\colon G \to A$ is a 1-cocycle if and only if $\alpha^*\colon G \to GA$ given by $g^{\alpha^*} = gg^{\alpha}$ is a homomorphism, and that $\alpha \mapsto G^{\alpha^*}$ defines a bijection between $\mathrm{Z}^1(G, A)$ and the set of complements of A in GA. Observe that then $\ker\alpha = G^{\alpha^*} \cap G$. We can give the following characterization:

(1.9) THEOREM: *Let A be a nonabelian irreducible G-group. Then*

$$\mathrm{E}_G(A) = \cap\{\mathrm{C}_G(B) \mid B \sim_G A\}.$$

Proof: By (1.3) we have that

$$\bigcap\{\mathrm{C}_G(B) \mid B \sim_G A\} = \bigcap\{\mathrm{C}_G(A_\alpha) \mid \alpha \in \mathrm{Z}^1(G, A)\}.$$

Consider the semidirect product GA. From the remark above this theorem we have immediately that

$$\mathrm{E}_G(A) = \bigcap\{H \mid H \text{ is a complement of } A \text{ in } GA\}.$$

In particular $\mathrm{E}_G(A) \trianglelefteq GA$ and $\mathrm{E}_G(A) \cap A = 1$. As $\mathrm{E}_G(A) \leq G$, we have that $\mathrm{E}_G(A) \leq \mathrm{C}_G(A)$. On the other hand, if $\alpha \in \mathrm{Z}^1(G, A)$ and $g \in \ker\alpha$, then $g \in \mathrm{C}_G(A)$ if and only if $g \in \mathrm{C}_G(A_\alpha)$. So we have that

$$\mathrm{E}_G(A) \leq \bigcap\{\mathrm{C}_G(A_\alpha) \mid \alpha \in \mathrm{Z}^1(G, A)\}.$$

Assume now that $g \in \bigcap\{\mathrm{C}_G(A_\alpha) \mid \alpha \in \mathrm{Z}^1(G, A)\}$. Then $a^{gg^{\alpha}} = a \ \forall a \in A$, $\forall \alpha \in \mathrm{Z}^1(G, A)$. With $\alpha = 0$ we obtain that in particular $a^g = a \ \forall a \in A$. Therefore $a^{g^{\alpha}} = a \ \forall a \in A$. Then $g^{\alpha} \in \mathrm{Z}(A) = 1$ as A is semisimple. Hence $g \in \mathrm{E}_G(A)$. ∎

(1.10) LEMMA: *Let A be an irreducible G-group such that $\mathrm{C}_G(A) < \mathrm{I}_G(A)$. Then $\mathrm{D}_G(A) \leq \mathrm{C}_G(A)$ if and only if $\mathrm{I}_G(A)/\mathrm{C}_G(A) \cong_G A$.*

Proof: Put $I = \mathrm{I}_G(A)$, etc. If $I/C \cong_G A$, as I/C is nonabelian, then it is not frattini, hence $D \leq C$, by the definition of $\mathrm{D}_G(A)$.

Assume now that $D \leq C$. Then $I \neq C$, as A is nonabelian. As I/D is a completely reducible G-group, we have $I/C \cong_G A$. ∎

(1.11) COROLLARY: *Let A be a nonabelian irreducible G-group such that $\{B \in \mathcal{CF}(G) \mid B \sim_G A\} \neq \emptyset$. Then*

$$\mathrm{D}_G(A) = \bigcap\{\mathrm{C}_G(B) \mid B \sim_G A, B \in \mathcal{CF}(G)\}.$$

(1.12) Definition: Let A be a nonabelian irreducible G-group. We set

$$\mathrm{J}_G(A) = \bigcap \{\mathrm{C}_G(B) \mid B \sim_G A,\ B \not\cong_G F,\ F \in \mathcal{CF}(G)\}$$

if $\{B \mid B \sim_G A,\ B \not\cong_G F,\ F \in \mathcal{CF}(G)\} \neq \emptyset$ and we put $\mathrm{J}_G(A) = \mathrm{I}_G(A)$ otherwise.

(1.13) Proposition: *Let A be a nonabelian irreducible G-group. Then*

$$\mathrm{I}_G(A) = \mathrm{J}_G(A)\,\mathrm{D}_G(A) \qquad \text{and} \qquad \mathrm{J}_G(A) \cap \mathrm{D}_G(A) = \mathrm{E}_G(A).$$

Proof: It is clear that $\mathrm{J}_G(A) \cap \mathrm{D}_G(A) = \mathrm{E}_G(A)$. Let $B \sim_G A$ such that $B \not\cong_G F$ if $F \in \mathcal{CF}(G)$ and set $S = \mathrm{C}_G(B)$. Put $I = \mathrm{I}_G(A)$, etc. We have that I/S is G-isomorphic to a proper subgroup of B. Then, as DS is a normal G-subgroup of I and I/D is a completely reducible G-group with its irreducible components G-equivalent to A, we have $DS = I$.

Assume that $DJ < I$. Let I/R be a chief factor of G such that $DJ \leq R$. Then $I/R \cong A$, because $D \leq R$. As $J \leq R$, there exists $B \sim_G A$, $B \not\cong_G F$ if $F \in \mathcal{CF}(G)$, such that $I/\mathrm{C}_G(B)$ has a factor isomorphic to I/R, in contradiction to $|I/\mathrm{C}_G(B)| < |A|$. ∎

2. On complemented chief factors

We say that a chief factor of G is a c-factor, resp. m-factor, if it is complemented in G by a subgroup, resp. maximal subgroup, of G; otherwise we say that it is a c′-factor, resp. m′-factor. Observe that an abelian chief factor is an m-factor, resp. m′-factor, if and only if it is a c-factor, resp. frattini.

As a consequence of [8], if $H, K \trianglelefteq G$, $H \leq K$, given two chief series of a group G

$$H = X_0 < X_1 < \cdots < X_n = K,$$
$$H = Y_0 < Y_1 < \cdots < Y_m = K,$$

between H and K, then $n = m$ and there exists a unique permutation π in $\mathfrak{S}_n$ such that:

(1) $X_i/X_{i-1} \sim_G Y_{i^\pi}/Y_{i^\pi-1}$.
(2) X_i/X_{i-1} and $Y_{i^\pi}/Y_{i^\pi-1}$ are simultaneously m-factors or m′-factors.
(3) If X_i/X_{i-1} and $Y_{i^\pi}/Y_{i^\pi-1}$ are m-factors, they have a maximal subgroup of G as common complement.

In particular, the number of m-factors in any chief series of G is the same. But this is no longer true for c-factors in spite of the equivalence between (3) and (4) in (1.4).

If A^*/A and B^*/B are chief factors of G, we put $B^*/B \searrow A^*/A$ if $A^*B = B^*$ and $A^* \cap B = A$.

(2.1) LEMMA: *Assume that B^*/B is a c′-factor and A^*/A is a c-factor of G such that $B^*/B \searrow A^*/A$, and that they are nonabelian. Let $I = \mathrm{I}_G(A^*/A)$ and $C = \mathrm{C}_G(A^*/A)$. Then there exists a normal subgroup X of G, $X \leq I$, such that, with $N = X \cap C$, one has $I/C \searrow B^*/B$, $X/N \searrow A^*/A$ and:*

(1) *$I/C \searrow X/N$, I/C is a c′-factor and X/N is a c-factor.*
(2) *G/C is a primitive group of type II and $S(G/C) = I/C$.*
(3) *There exists a supplement F of I/C in G such that G/N is isomorphic to the natural semidirect product of F/C by I/C.*

Proof: Observe that $I/C \searrow B^*/B$, hence I/C is a c′-factor. Take a chief series between A and C. By multiplying by A^* there results another one between A^* and I. The formed transoms are chief factors of G. As I/C is a c′-factor but A^*/A is a c-factor, we obtain chief factors M/X and Y/N such that

$$I/C \searrow M/Y \searrow X/N \searrow A^*/A,$$

M/Y is a c′-factor and X/N is a c-factor. We may assume that $N = 1$.

Let U be a complement of X in G. Consider $K = U \cap C$. X centralizes and U normalizes K, hence $K \trianglelefteq G$. We have $U(KX) = G$ and $U \cap KX = K(U \cap X) = K$, hence KX/K is a c-factor. As $I/C \searrow KX/K$, we may assume that $K = 1$.

Observe finally that $ux \mapsto (uC)(xC)$, where $u \in U$ and $x \in X$, defines an isomorphism between $G = UX$ and the natural semidirect product $(UC/C)[I/C]$. ∎

We can construct the situation of the lemma, similarly to an unpublished example of Förster, which turns out to be characteristic of this context. Let G_0 be a group, $X_0 \trianglelefteq G_0$ and let H_0 be a supplement of X_0 in G_0. Set $L_0 = H_0 \cap X_0$.

Consider the semidirect product

$$G = H_0[X_0] = \{\,(h,x) \mid h \in H_0,\ x \in X_0\,\}.$$

Put $H = \{\,(h,1) \mid h \in H_0\,\}$, $X = \{\,(1,x) \mid x \in X_0\,\}$. We have that $G = HX$, $H \cap X = 1$, $X \trianglelefteq G$.

Let now $B = \{\,(d,1) \mid d \in L_0\,\}$, $L = \{\,(1,d) \mid d \in L_0\,\}$. We have that $C = \{\,(d,d^{-1}) \mid d \in L_0\,\}$ verifies:

(1) $C \trianglelefteq G$, $X \cap C = 1$, $I := XC = XB$, $B = H \cap I$.
(2) $X \cap BC = L$, $BL = BC$, $L \trianglelefteq HC$, $[L,C] = 1$.

(3) $L \cong_H B \cong_H C$.

Assume now that G_0 is a primitive group of type II and that $S(G_0) = X_0$ is not complemented in G_0 ($G_0 = \mathrm{Aut}\, A_6$, $X_0 = \mathrm{Inn}\, A_6$, for instance). Then we are in a situation like the lemma.

(2.2) PROPOSITION: *Assume that, in the above situation, X_i/X_{i-1} and $Y_{i^\pi}/Y_{i^\pi-1}$ are m′-factors. Then, either*

(a) *both factors are c′-factors, or*

(b) *both factors are nonabelian c-factors, or*

(c) *both factors are nonabelian, one of them is a c-factor, the other one is a c′-factor and there exist normal subgroups I, C, X and N of G verifying (1)–(3) of the lemma.*

Proof: Assume that X_i/X_{i-1} is a c′-factor and $Y_{i^\pi}/Y_{i^\pi-1}$ is a c-factor. Then, as we see in [8], either there exist R^*, R, normal subgroups of G with

$$R^*/R \searrow X_i/X_{i-1}, \qquad R^*/R \searrow Y_{i^\pi}/Y_{i^\pi-1},$$

or there exist S^*, S, normal subgroups of G with

$$X_i/X_{i-1} \searrow S^*/S, \qquad Y_{i^\pi}/Y_{i^\pi-1} \searrow S^*/S,$$

where R^*/R and S^*/S are m′-factors, or there exist S^*, S, Z, T normal subgroups of G with

$$X_i/X_{i-1} \searrow S^*/Z \searrow T/S, \qquad Y_{i^\pi}/Y_{i^\pi-1} \searrow S^*/T \searrow Z/S,$$

where S^*/Z and S^*/T are m′-factors, and Z/S and T/S are m-factors (this is called an m-crossing in [8]). In the first cases we have that R^*/R must be a c′-factor, whereas S^*/S must be a c-factor. In the last case, the m-factors in the crossing Z/S and T/S have a maximal subgroup as common complement, hence $G/Z \cong G/T$. On the other hand, $Y_{i^\pi}/Y_{i^\pi-1}$ is a c-factor, then so is S^*/T. Therefore also S^*/Z is a c-factor, whereas X_i/X_{i-1} is a c′-factor. We are then in either case in the situation of the lemma. ∎

(2.3) Definition: Let A be an irreducible G-group. We say that A is of cc′-type in G if there exist two chief series of G in which one has the case (c) of the precedent proposition with $A \sim_G X_i/X_{i-1}$. (Clearly this forces A to be nonabelian.)

(2.4) PROPOSITION: *Let v be the number of equivalence classes of irreducible G-groups of cc'-type. Then the number of complemented chief factors on two chief series of G differs by at most v.*

Proof: It is a consequence of the proposition, as on a chief series of G for each nonabelian crown there is at most one m′-factor. If the crown corresponds to a factor of cc′-type, this shows that on each chief series there is at most one c′-factor corresponding to the crown. ∎

(2.5) THEOREM: *Let A be a nonabelian irreducible G-group. Then A is of cc'-type in G if and only if*

$$\mathrm{E}_G(A) < \mathrm{D}_G(A) < \mathrm{I}_G(A)$$

and $S(P)$ is a c'-factor of P, where P is the corresponding primitive epimorphic image of G.

Proof: Put $E = \mathrm{E}_G(A)$, etc. Assume that $E < D < I$. Then there exist normal subgroups R and H of G such that $I/R \sim_G A$, $H = \mathrm{C}_G(B)$, $B \sim_G A$, $B \not\cong_G F$ if $F \in \mathcal{CF}(G)$, $H < I$. Put $K = H \cap R$. Then we have $A \sim_G H/K$. By (1.3), H/K is a c-factor of G. As I/R is a c′-factor we are with a cc′-situation.

Conversely, if A is of cc′-type, we obtain normal subgroups I, R, H and K of G and a subgroup U of G such that $I/R \sim_G A$, I/R and H/K are m′-factors, I/R is a c′-factor, $I/R \searrow H/K$ and U complements H/K in G. Observe that $R = \mathrm{C}_G(I/R) = \mathrm{C}_G(H/K)$. As in the proof of (1.3) we obtain a G-group B, $B \sim_G A$, such that

$$\mathrm{C}_G(B) = H\,\mathrm{C}_U(H/K) = H(U \cap R) = HK = H.$$

Suppose now that there exists $F \in \mathcal{CF}(G)$ such that $F \cong_G B$. Then $\mathrm{I}_G(F) = \mathrm{I}_G(B) = I$ and $\mathrm{C}_G(F) = \mathrm{C}_G(B) = H$, hence $F \cong_G I/H$. Therefore $I/H \sim_G I/R$, hence G/K is a primitive group of type III and H/K is an m-factor, contrary to the hypothesis. And we have that $\mathrm{D}_G(A) \le R < I$, $\mathrm{J}_G(A) \le H < I$. ∎

(2.6) THEOREM: *Let G a primitive group of type II and $M = S(G)$. Let $M = T_1 \times \cdots \times T_n$, where $T_i \cong T$, $1 \le i \le n$, $T = T_1$ a simple group. Put $N = \mathrm{N}_G(T)$, $I = \mathrm{I}_G(T)$, $C = \mathrm{C}_G(T)$. Then the following assertions are equivalent:*

1. *M is a c-factor of G,*
2. *either I/C is a c-factor of N, or I/C is a c'-factor of N and there exists an N-group $S \sim_N T$ such that $M \le \mathrm{C}_N(S) < I$.*

Proof: Put $K = T_2 \times \cdots \times T_n$. By [4], M is a c-factor of G if and only if M/K is a c-factor of N. By (1.3), M/K is a c-factor if and only if there exists an

N-group $S \sim_N T$ such that $M \leq \mathrm{C}_N(S)$. Observe that then $S \sim_N T$ and that, if I/C is a c′-factor, then $\mathrm{C}_N(S) < I$. ∎

References

[1] M. Aschbacher and R. Guralnick, *Some applications of the first cohomology group,* Journal of Algebra **90** (1984), 446–460.

[2] P. Förster, *Chief factors, crowns, and the generalised Jordan–Hölder Theorem,* Communications in Algebra **16** (1988), 1627–1638.

[3] W. Gaschütz, *Praefrattinigruppen,* Archiv der Mathematik **13** (1962), 418–426.

[4] F. Gross and L. G. Kovács, *On normal subgroups which are direct products,* Journal of Algebra **90** (1984), 133–168.

[5] K. W. Gruenberg, *Cohomological Topics in Group Theory,* Lecture Notes in Mathematics **143**, Springer-Verlag, Berlin, 1970.

[6] K. W. Gruenberg, *Groups of non-zero presentation rank,* Symposia Mathematica **17** (1976), 215–224.

[7] J. P. Lafuente, *Nonabelian crowns and Schunck classes of finite groups,* Archiv der Mathematik **42** (1984), 32–39.

[8] J. P. Lafuente, *Maximal subgroups and the Jordan–Hölder Theorem,* Journal of the Australian Mathematical Society **46** (1989), 356–364.

[9] J. P. Lafuente, *On the second Loewy term of projectives of a group algebra,* Israel Journal of Mathematics **67** (1989), 170–180.

[10] H. Neumann, *Varieties of Groups,* Springer-Verlag, Berlin, 1967.

[11] D. J. S. Robinson, *A Course in the Theory of Groups,* Springer-Verlag, New York, 1995.

[12] J. P. Serre, *Cohomologie Galoisienne,* Lecture Notes in Mathematics **5**, Springer-Verlag, Berlin, 1964.